NUM. II.

Ejuſdem

THEOREMATA
circa reductionem formularum integralium ad quadraturam circuli.

Lemma I.

1. IN circulo cujus radius eſt $= 1$, & ſemiperipheria $= \pi$ ſit anguli cujusvis s ſinus $= x$, erit eadem quantitas x ſinus omnium arcuum in hac ſerie infinita contentorum:

s; $\pi - s$; $2\pi + s$; $3\pi - s$; $4\pi + s$; $5\pi - s$; &c.
Præterea vero x erit ſinus omnium arcuum negativorum in hac ſine contentorum.

$-\pi - s$; $-2\pi + s$; $-3\pi - s$; $-4\pi + s$; $-5\pi - s$; &c.

Corollarium I.

2. Si igitur i denotet numerum quemcunque integrum affirmativum, erit tam arcuum omnium in hac expreſſione contentorum $\pm 2i\pi + s$, quam arcuum in hac expreſſione $\pm (2i + 1)\pi - s$ contentorum idem ſinus communis x.

Coroll. 2.

3. Cum angulorum negative ſumtorum ſinus fiant negativi, erit angulorum in hac forma $\pm 2i\pi - s$ contentorum ſinus $= -x$; angulorumque in hac forma, $\pm (2i + 1)\pi + s$ contentorum ſinus $= -x$ ſiquidem anguli s fuerit ſinus $= + x$.

Lemma 2.

4. In circulo cujus radius $= 1$, & ſemiperipheria $= \pi$, ſit anguli cujusvis s coſinus $= y$, erit eadem quantitas y coſinus omnium angulorum affirmativorum in hac ſerie contentorum.

$s;\ 2\pi - s;\ 2\pi + s;\ 4\pi - s;\ 4\pi + s;\ 6\pi - s;$ &c.

pariterque eadem quantitas y erit coſinus omnium angulorum negativorum in hac ſerie contentorm.

$-s; 2\pi + s;\ -2\pi - s;\ -4\pi + s;\ -4\pi - s;\ -6\pi + s;$ &c.

Coroll. 1.

5. Si igitur i denotet numerum quemcunque integrum affirmativum, erit omnium arcuum in hac expreſſione generali $\pm 2i\pi \pm s$ contentorum idem communis coſinus $= y$.

Coroll. 2.

6. Quoniam anguli duobus rectis ſeu arcu π aucti ſive minuti coſinus fit negativus, erit omnium angulorum in hac forma $\pm (2i + 1)\pi \pm s$ idem coſinus $= -y$, ſiquidem anguli s coſinus fuerit $= + y$.

Lemma 3.

7. Iisdem poſitis, ſi t ſit tangens anguli s, erit quoque t tangens omnium angulorum tam affirmativorum quam negativorum in his duabus ſeriebus contentorum.

$s;\ \pi + s;\ 2\pi + s;\ 3\pi + s;\ 4\pi + s;\ 5\pi + s;$ &c.

$-\pi + s;\ -2\pi + s; -3\pi + s;\ -4\pi + s;\ -5\pi + s;$ &c.

Coroll.

8. Denotante ergo i numerum quemcunque affirmativum, erit omnium angulorum in hac expreſſione $\pm i\pi + s$

conten-

contentorum eadem communis tangens t; angulorum vero $\pm i\pi - s$ tangens erit $= - t$; ſiquidem anguli s tangens ſit $= + t$.

Problema.

9. Invenire radices hujus æquationis infinitæ

$$x = z - \frac{z^3}{1.2.3.} + \frac{z^5}{1.2.3.4.5.} - \frac{z^7}{1.2.3.\ldots.7} + \&c.$$

Solutio.

Si z denotet arcum circuli radii $= 1$, erit x ipſius ſinus. Ponamus ergo arcum, cujus ſinus ſit $= x$, eſſe s, erunt infiniti ipſius z valores in his duabus ſeriebus contenti.

$$s;\ \pi - s;\ 2\pi + s;\ 3\pi - s;\ 4\pi + s;\ 5\pi - s;\ \&c.$$

$$-\pi - s;\ -2\pi + s;\ -3\pi - s;\ -4\pi + s;\ -5\pi - s;\ \&c.$$

Q. E. J.

Coroll. 1.

10. Si igitur æquatio propoſita in hanc formam transmutetur.

$$0 = 1 - \frac{z}{1.x} + \frac{z^3}{1.2.3.x} - \frac{z^5}{1.2.3.4.5\,x} + \frac{z^7}{1.2.\ldots.7.x} - \&c.$$

ejus habebuntur factores numero infiniti ſequentes

$$\left(1 - \frac{z}{s}\right)\left(1 - \frac{z}{\pi - s}\right)\left(1 + \frac{z}{\pi + s}\right)\left(1 - \frac{z}{2\pi + s}\right)\left(1 + \frac{z}{2\pi - s}\right)\left(1 - \frac{z}{3\pi - s}\right)\left(1 + \frac{z}{3\pi + s}\right)$$

&c. in quibus factoribus lex progreſſionis facile perſpicitur.

Coroll. 2.

11. Cum igitur coefficiens termini ſecundi in producto æquetur ſummæ coefficientium ipſius z in omnibus factoribus erit.

$$\frac{1}{x} = \frac{1}{s} + \frac{1}{\pi - s} - \frac{1}{\pi + s} - \frac{1}{2\pi - s} + \frac{1}{2\pi + s} + \frac{1}{3\pi - s} - \&c.$$

Co-

Coroll. 3.

12. Sit $s = \frac{m}{n}\pi$; ita ut fiat $x =$ ſin A. $\frac{m}{n}\pi$ erit ſerie per $\frac{\pi}{n}$ multiplicata; $\frac{\pi}{nx} =$

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \&c.$$

Coroll. 4.

13. Quoniam coefficiens ipſius z^2 in producto, qui eſt $= 0$, æquatur ſummæ factorum ex binis coefficientibus ipſius z in factoribus, hæc vero ſumma biſſumta æqualis eſt quadrato ſummæ iſtorum coefficientium demta ſumma quadratorum eorundem, erit:

$$\frac{1}{xx} = \frac{1}{ss} + \frac{1}{(\pi-s)^2} + \frac{1}{(\pi+s)^2} + \frac{1}{(2\pi-s)^2} + \frac{1}{(2\pi+s)^2} + \&c.$$

Coroll. 5.

14. Poſito ergo iterum $s = \frac{m}{n}\pi$, ut ſit $x =$ ſin A. $\frac{m\pi}{n}$, prodibit iſta ſummatio: $\frac{\pi\pi}{nnxx} =$

$$\frac{1}{m^2} + \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \&c.$$

Scholion.

15. Poſſem hoc modo ultra progredi, atque ſummas altiorum poteſtatum determinare, quoniam vero hoc alibi jam feci, atque ad inſtitutum noſtrum hæ ſeries ſufficiunt; ulteriori inveſtigationi ſuperſedebo.

Proble-

Problema 2.

16. Invenire radices hujus æquationis infinitæ $y = 1 - \frac{z^2}{1.2} + \frac{z^4}{1.2.3.4} - \frac{z^6}{1.2\ldots.6} + \frac{z^8}{1.2\ldots.8} - \&c.$

Solutio.

Si z denotet arcum in circulo cujus radius $= 1$, erit y hujus arcus coſinus. Quodſi ergo capiatur arcus s cujus coſinus ſit $= y$; continebuntur innumerabiles ipſius z valores in binis ſequentibus ſeriebus:

$+s;\ +2\pi - s;\ +2\pi + s;\ +4\pi - s;\ +4\pi + s;$ &c.

$-s;\ -2\pi + s;\ -2\pi - s;\ -4\pi + s;\ -4\pi - s;$ &c.

Q. E. J.

Coroll. 1.

17. Si igitur æquatio propoſita in hanc formam transmutetur:

$$0 = 1 - \frac{z^2}{1.2(1-y)} + \frac{z^4}{1.2.3.4.(1-y)} - \frac{z^6}{1.2\ldots.6(1-y)} + \&c.$$

ejus habebuntur factores numero infiniti ſequentes.

$$\left(1 - \frac{zz}{ss}\right)\left(1 - \frac{zz}{(2\pi-s)^2}\right)\left(1 - \frac{zz}{(2\pi+s)^2}\right)\left(1 - \frac{zz}{(4\pi-s)^2}\right)\left(1 - \frac{zz}{(4\pi+s)^2}\right)\&c.$$

Coroll. 2

18. Cum igitur in producto coefficiens ipſius zz æqualis ſit ſummæ coefficientium ipſius zz in factoribus habebitur ſequentis ſeriei ſummatio:

$$\frac{1}{2(1-y)} = \frac{1}{ss} + \frac{1}{(2\pi-s)^2} + \frac{1}{(2\pi+s)^2} + \frac{1}{(4\pi-s)^2} + \frac{1}{(4\pi+s)^2} + \&c.$$

Coroll. 3.

19. Ponatur $s = \frac{m}{n}\pi$, ut ſit $y = $ coſ. A. $\frac{m}{n}\pi$ & $1 - y$ $= 2\left(\text{ſin.}\ \frac{m}{2n}\pi\right)^2$; erit $\frac{\pi\pi}{2nn(1-y)} = \frac{\pi\pi}{4nn(\text{ſin A.}\frac{m}{2n}\pi)^2}$

$$= \frac{1}{mm} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n-m)^2} + \frac{1}{(4n+m)^2} + \&c.$$

quæ congruit cum §. 14. ſi modo hic loco $2n$ ſcribatur n,

Problema 3.

20. Invenire radices ipſius z hujus æquationis infinitæ:

$$t = \frac{z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2.\ldots 7} + \&c.}{1 - \frac{z^2}{1.2} + \frac{z^4}{1.2.3.4} - \frac{z^6}{1.2.\ldots 6} + \&c.}$$

Solutio.

Si z denotet arcum circuli, cujus radius $= 1$, erit t tangens hujus arcus, ſi ergo ſumatur arcus s in hoc circulo cujus tangens ſit $= t$, erunt infiniti ipſius z valores ſequentes.

s; $\pi + s$; $2\pi + s$; $3\pi + s$; $4\pi + s$; $5\pi + s$; &c.
$-\pi + s$; $-2\pi + s$; $-3\pi + s$; $-4\pi + s$; $-5\pi + s$; &c.

Q. E. J.

Coroll. 1.

21. Reducatur æquatio propoſita ad hanc formam

$$0 = 1 - \frac{z}{t} - \frac{z^2}{1.2} + \frac{z^3}{1.2.3.t} + \frac{z^4}{1.2.3.4} - \frac{z^5}{1.2.3.4.5t} + \&c.$$

ejusque

ejusque factores ſimplices erunt ſequentes.

$$\left(1-\frac{z}{s}\right)\left(1+\frac{z}{\pi-s}\right)\left(1-\frac{z}{\pi+s}\right)\left(1+\frac{z}{2\pi-s}\right)\left(1-\frac{z}{2\pi+s}\right)\left(1+\frac{z}{3\pi-s}\right)\&c$$

Coroll. 2.

22. Cum igitur coefficiens ipſius z in æquatione æqualis ſit ſummæ coefficientium ipſius z in ſingulis factoribus; erit

$$\frac{1}{t}=\frac{1}{s}-\frac{1}{\pi-s}+\frac{1}{\pi+s}-\frac{1}{2\pi-s}+\frac{1}{2\pi+s}-\frac{1}{3\pi-s}+$$

&c.

Coroll. 3.

23. Ponatur $s=\frac{m}{n}\pi$ ut ſit $t=$ tang. A $\frac{m}{n}\pi$ $=$ $\frac{x}{y}$ ſi ſit $x=$ ſin. A $\frac{m}{n}\pi$ & $y=$ coſ. A$\frac{m}{n}\pi$; atque prodibit ſequentis ſeriei ſummatio: $\frac{\pi}{nt}=\frac{\pi y}{nx}=$

$$\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\&c.$$

Coroll. 4.

24. Summa quadratorum ſingulorum terminorum ſeriei (22) æqualis eſt quadrato ſummæ ipſorum $\frac{1}{tt}$ demta duplici ſumma factorum ex binis, hoc eſt -1 erit ergo ſumma quadratorum $=\frac{1}{tt}+1=\frac{yy}{xx}+1=\frac{1}{xx}$. Prodibit ergo uti in §. 13. hæc ſummatio.

$$\frac{1}{xx}=\frac{1}{ss}+\frac{1}{(\pi-s)^2}+\frac{1}{(\pi+s)^2}+\frac{1}{(2\pi-s)^2}+\frac{1}{(2\pi+s)^2}$$

$+$ &c.

 Theore-

THEOREMA. I.

25. Denonante π ſemiperipheriam circuli, cujus radius $= 1$, ſi fuerit $x =$ ſin A. $\frac{m}{n}\pi$; erit $\frac{\pi(p+qy)}{nx} = \frac{p+q}{m} + \frac{p-q}{n-m} - \frac{p+q}{n+m} - \frac{p-q}{2n-m} + \frac{p+q}{2n+m} + \frac{p-q}{3n-m} - \frac{p+q}{3n+m} -$ &c.

Demonſtratio.

Si ſeries §. 12. inventa multiplicetur per p erit

$$\frac{\pi p}{nx} = \frac{p}{m} + \frac{p}{n-m} - \frac{p}{n+m} - \frac{p}{2n-m} + \frac{p}{2n+m} + \text{\&c.}$$

Et, ſi ſeries §. 23. multiplicetur per q erit

$$\frac{\pi qy}{nx} = \frac{q}{n} - \frac{q}{n-m} + \frac{q}{n+m} - \frac{q}{2n-m} + \frac{q}{2n+m} - \text{\&c.}$$

Addantur hæ duæ ſeries, prodibitquæ ſeries propoſita, cujus propterea ſumma eſt $= \frac{\pi(p+qy)}{nx}$. Q. E. D.

Coroll. I.

26. ſumatur $p = q$, prodibit iſta ſummatio:

$$\frac{\pi(1+y)}{2nx} = \frac{1}{m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{4n-m} + \frac{1}{4n+m} - \text{\&c.}$$

at eſt $\frac{x}{1+y}$ tang. $\frac{m}{2n}\pi$, quæ ergo poſito n loco $2n$ congruit cum ſerie §. 23. inventa.

Coroll. 2.

27. Si $q = -p$; prodit hæc ſummatio:

$$\frac{\pi(1-y)}{2nx} = \frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \text{\&c.}$$

at eſt

at eſt $\frac{1-y}{x} = \text{tang. A.}\ \frac{m}{2n}\pi$; hincque $\frac{x}{1-y} = \text{tang. A.}$ $\left(\frac{-m}{2n}\pi + \tfrac{1}{2}\pi\right) = \text{tang. A.}\ \frac{(n-m)}{2n}\pi$, quæ ſeries denuo ſcripto m loco $n-m$ ad præcedentem reducitur.

Problema 4.

28. Invenire ſummam hujus ſeriei:

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} -$$

&c. per formulam integralem.

Solutio.

Tribuantur ſingulis fractionibus numeratores, qui ſint poteſtates ipſius z, quarum exponentes denominatoribus æquentur; habebiturque hæc ſeries.

$$\frac{z^m}{m} + \frac{z^{n-m}}{n-m} - \frac{z^{n+m}}{n+m} - \frac{z^{2n-m}}{2n-m} + \frac{z^{2n+m}}{2n+m} + \&c.$$

quæ facto $z = 1$ in illam abibit. Ponatur ſumma hujus ſeriei $= s$; ac ſumtis differentialibus erit

$$\frac{z\,ds}{dz} = z^m + z^{n-m} - z^{n+m} - z^{2n-m} + z^{2n+m} + z^{3n-m} -$$

&c. quæ ſeries compoſita eſt ex duabus geometricis eritque ideo ejus ſumma $= \frac{z^m}{1+z^n} + \frac{z^{n-m}}{1+z^n}$. Habemus ergo

$$\frac{z\,ds}{dz} = \frac{z^m + z^{n-m}}{1+z^n} \ \& \ ds = \frac{z^{m-1} + z^{n-m-1}}{1+z^n}\,dz$$

Conſequenter $s = \int \frac{z^{m-1} + z^{n-m-1}}{1+z^n}\,dz$, hujus itaque

 inte-

integralis valor caſu quo $z = 1$, dabit ſummam ſeriei propoſitæ. Q. E. J.

Coroll. 1.

29. Quoniam igitur ſeriei propoſitæ ſumma eſt

$$\frac{\pi}{nx} = \frac{\pi}{n\,\text{ſin. A.}\,\frac{m}{n}\pi};\ \text{erit}\ \frac{\pi}{n\,\text{ſin. A.}\,\frac{m}{n}\pi} = \int\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz$$

ſi poſt integrationem ponatur $z = 1$. Hoc ergo caſu integrale formulæ $\int\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz$ ope circuli poteſt exhiberi.

Coroll. 2.

30. Ponendo ergo loco m & n numeros definitos, habebuntur ſequentes integrationes caſu $z = 1$:

ſi $m = 1$; $n = 2$ erit $\frac{\pi}{2} = \int\frac{2\,dz}{1+zz}$

ſi $m = 1$; $n = 3$ erit $\frac{2\pi}{3\sqrt{3}} = \int\frac{1+z}{1+z^3}dz = \int\frac{dz}{1-z+zz}$

ſi $m = 1$; $n = 4$ erit $\frac{\pi}{2\sqrt{2}} = \int\frac{1+zz}{1+z^4}dz$,

ſi $m = 1$; $n = 6$ erit $\frac{\pi}{3} = \int\frac{1+z^4}{1+z^6}dz$,

quæ omnia integratione actu inſtituta veritati conſentanea deprehenduntur.

Problema. 5.

31. Invenire ſummam hujus ſeriei:

$\frac{1}{m}$

$\frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \frac{1}{3n+m}$ — &c. per formulam integralem.

Solutio.

Conjungantur cum his fractionibus numeratores idonei, ut ante fecimus, ac ponatur:

$$s = \frac{z^{m}}{m} - \frac{z^{n-m}}{n-m} + \frac{z^{n+m}}{n+m} - \frac{z^{2n-m}}{2n-m} + \frac{z^{2n+m}}{2n+m} - \&c.$$

erit utique facto $z = 1$, valor ipſius s ſumma ſeriei propoſitæ. Inſtituatur jam differentiatio, ac prodibit

$$\frac{z\,ds}{dz} = z^{m} - z^{n-m} + z^{n+m} - z^{2n-m} + z^{2n+m} - \&c.$$

cujus ſeriei ſumma quia exhiberi poteſt, habebitur.

$$\frac{z\,ds}{dz} = \frac{z^{m} - z^{n-m}}{1-z^{n}},$$ unde fit $s = \int \frac{z^{m-1} - z^{n-m-1}}{1-z^{n}}\,dz$

Hujus ergo formulæ integralis valor caſu $z = 1$ dabit ſummam ſeriei propoſitæ. Q. E. J.

Coroll. 1.

32. Per §. 23. eſt ſeriei propoſitæ ſumma $=$

$$\frac{\pi}{nt} = \frac{\pi y}{nx} = \frac{\pi \text{ coſ. A.} \frac{m}{n}\pi}{n \text{ ſin A.} \frac{m}{n}\pi}.$$ Quamobrem erit

$$\frac{\pi \text{ coſ. A} \frac{m}{n}\pi}{n \text{ ſin. A.} \frac{m}{n}\pi} = \int \frac{z^{m-1} - z^{n-m-1}}{1-z^{n}}\,dz$$ caſu quo poſt integrationem ponitur $z = 1$.

Coroll. 2.

33. Caſus ergo ſimpliciores ita ſe habebunt.

ſi $m = 1,\ n = 3$ erit $\frac{\pi}{3\sqrt{3}} = \int \frac{(1-z)dz}{1-z^3} = \int \frac{dz}{1+z+zz}$

ſi $m = 1,\ n = 4$ erit $\frac{\pi}{4} = \int \frac{(1-z^2)dz}{1-z^4} = \int \frac{dz}{1+z^2}$

ſi $m = 1;\ n = 6$ erit $\frac{\pi}{2\sqrt{3}} = \int \frac{(1-z^4)dz}{1-z^6} = \int \frac{(1+zz)\,dz}{1+zz+z^4}$

quarum formularum congruentia cum veritate poſt integrationem actu inſtitutam ſponte patet.

Scholion.

34. Si in binis ſeriebus nunc tractatis bini termini in unum colligantur, orientur ſequentes ſummationes.

$$\frac{\pi}{n \,\text{ſin A.}\, \frac{m}{n}\pi} = \frac{1}{m} + \frac{2m}{n^2-m^2} - \frac{2m}{4n^2-m^2} + \frac{2m}{9n^2-m^2} - \frac{2m}{16n^2-m^2}$$

$+$ &c.

$$\frac{\pi \,\text{coſ.A.}\, \frac{m}{n}\pi}{n \,\text{ſin. A.}\, \frac{m}{n}\pi} = \frac{1}{m} - \frac{2m}{n^2-m^2} - \frac{2m}{4n^2-m^2} - \frac{2m}{9n^2-m^2} - \frac{2m}{16n^2-m^2}$$

$-$ &c.

His ergo ordinatis habebimus duas ſequentes ſeries notatu, maxime dignas:

$$\frac{\pi}{2mn \,\text{ſin A.}\, \frac{m}{n}\pi} - \frac{1}{2mm} = \frac{1}{n^2-m^2} - \frac{1}{4n^2-m^2} + \frac{1}{9n^2-m^2} - \frac{1}{16n^2-m^2}$$

$+$ &c.

$\frac{1}{2mm}$

$$\frac{1}{2mm}-\frac{\pi\,\text{coſ.A.}\frac{m}{n}\pi}{2nm\,\text{ſin A.}\frac{m}{n}\pi}=\frac{1}{n^2-m^2}+\frac{1}{4n^2-m^2}+\frac{1}{9n^2-m^2}+$$

$\frac{1}{16n^2-m^2}+$ &c. quæ ſi invicem addantur dabunt:

$$\frac{\pi(1-\text{coſ. A.}\frac{m}{n}\pi)}{4mn\,\text{ſin. A.}\frac{m}{n}\pi}=\frac{\pi\,\text{coſ. A.}\frac{m}{2n}\pi}{4mn\,\text{ſin. A.}\frac{m}{2n}\pi}=$$

$$\frac{1}{n^2-m^2}+\frac{1}{9n^2-m^2}+\frac{1}{25n^2-m^2}+\frac{1}{49n^2-m^2}+\text{ \&c.}$$

Hasque ſeries ante aliquot annos ex principiis longe diverſis ſum conſecutus.

THEOREMA. 2.

35. Summa hujus ſeriei quadratorum.

$$\frac{1}{m^2}-\frac{1}{(n-m)^2}-\frac{1}{(n+m)^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}-\frac{1}{(3n-m)^2}$$
$-$ &c.

eſt $=\dfrac{\pi^2\,\text{coſ. A.}\frac{m}{n}\pi}{nn\,(\text{ſin. A.}\frac{m}{n}\pi)^2}$

Demonſtratio.

In §. 12. vidimus eſſe $\dfrac{\pi}{n\,\text{ſin. A.}\frac{m}{n}\pi}=\dfrac{1}{m}+\dfrac{1}{n-m}$

$$-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\frac{1}{3n+m}-\text{ \&c.}$$

quæ

quæ æqualitas cum ſemper habeat locum, quicquid ſit m, differentialia quoque æqualia eſſe debent. Differentiemus ergo tam ſeriem, quam ipſius ſummam, poſita m variabili, eruntque differentialia quoque æqualia. Habebitur autem utrinque per dm diviſo:

$$\frac{-\pi^2 \operatorname{coſ.A.}\frac{m}{n}\pi}{nn\,(\operatorname{ſin.A.}\frac{m}{n}\pi)^2} = \frac{-1}{m^2} + \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} - \frac{1}{(2n-m)^2} - \frac{1}{(2n+m)^2}$$

$+$ &c.

Mutentur ſigna, atque habebitur ſumma ſeriei quadratorum propoſitæ. Q. E. J.

Coroll. 1.

36. Evolvamus aliquot caſus ſimpliciores, ſitque

$m = 1$; $n = 2$; erit ſin A. $\frac{m}{n}\pi = 1$ & coſ. A. $\frac{m}{n}\pi = 0$, unde

$0 = 1 - 1 - \frac{1}{9} + \frac{1}{9} + \frac{1}{25} - \frac{1}{25} - \frac{1}{49} +$ &c.

ſit $m = 1$; $n = 3$; erit ſin A. $\frac{m}{n}\pi = \frac{\sqrt{3}}{2}$ & coſ. A. $\frac{m}{n}\pi = \frac{1}{2}$ unde

$\frac{2\pi\pi}{27} = 1 - \frac{1}{4} - \frac{1}{16} + \frac{1}{25} + \frac{1}{49} - \frac{1}{64} - \frac{1}{100} +$ &c.

ſit $m = 1$; $n = 4$ erit ſin A. $\frac{m}{n}\pi =$ coſ. A. $\frac{m}{n}\pi = \frac{1}{\sqrt{2}}$ unde:

$\frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{9} - \frac{1}{25} + \frac{1}{49} + \frac{1}{81} - \frac{1}{121} - \frac{1}{169} +$ &c.

Coroll. 2.

37. Multiplicemus ſeriem $\frac{2\pi\pi}{27} = 1 - \frac{1}{4} - \frac{1}{16} + \frac{1}{25} + \frac{1}{49} - \frac{1}{64} - \frac{1}{100} +$ &c. in qua quadrata per ternarium divi-

diviſibilia deſunt, per $\frac{9}{8} = 1 + \frac{1}{9} + \frac{1}{81} + \frac{1}{729} +$ &c. ut omnia quadrata occurrant; eritque $\frac{\pi\pi}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} -$ &c. cujus veritas jam alibi a me eſt demonſtrata.

Coroll. 3.

38. Cum ſit ex §. 14. $\frac{\pi\pi}{nnxx} = \frac{\pi\ \pi}{nn(\text{ſin.}\frac{m}{n}\pi)^2} =$

$$\frac{1}{m^2} + \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \&c.$$

erit his ſeriebus additis $\frac{\pi\pi(1 + \text{coſ. A}\frac{m}{n}\pi)}{2nn(\text{ſin.}\frac{m}{n}\pi)^2} =$

$$\frac{1}{m^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n-m)^2} + \frac{1}{(4n+m)^2} + \&c.$$

quæ ſeries ſcribendo n loco $2n$ ad illam reducitur, eſt enim $\frac{1 + \text{coſ. A}\frac{m}{n}\pi}{(\text{ſin.A}\frac{m}{n}\pi)^2} = \frac{1}{2(\text{ſin. A}\frac{m}{2n}\pi)^2}$.

Scholion.

39. Summatio ergo in hac propoſitione demonſtrata directe deduci potuiſſet ex ſummatione ſeriei §. 14. datæ. Cum enim ſit

$$\frac{\pi\ \pi}{nn(\text{ſin.A}\frac{m}{n}\pi)^2} = \frac{1}{m^2} + \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \&c.$$

erit

erit quoque ſcribendo $2n$ loco n; $\frac{\pi\pi}{4nn(\text{ſin.}A\frac{m}{2n}\pi)^2} =$

$$\frac{\pi\pi(1+\text{coſ.}A.\frac{m}{n}\pi)}{2nn(\text{ſin.}A\frac{m}{n}\pi)^2} = \frac{1}{m^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n-m)^2}$$

$+ \frac{1}{(4n+m)^2} +$ &c. a cujus duplo ſi illa ſubtrahatur, remanebit propoſita

$$\frac{\pi\pi\,\text{coſ.}A\frac{m}{n}\pi}{nn(\text{ſin.}A\frac{m}{n}\pi)^2} = \frac{1}{m^2} - \frac{1}{(n-m)^2} - \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2}$$

$+ \frac{1}{(2n+m)^2} -$ &c.

ſimili autem modo ex ſerie §. 23, quæ ob ſigna alternantia maxime videtur regularis deduci poteſt ſeries §. 12. exhibita.

Cum enim ſit $\frac{\pi\,\text{coſ.}A.\frac{m}{n}\pi}{n\,\text{ſin.}A\frac{m}{n}\pi} =$

$$\frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{\&c.}$$

erit, ſi $2n$ loco n ſcribatur $\frac{\pi\,\text{coſ.}A\frac{m}{2n}\pi}{2n\,\text{ſin.}A\frac{m}{2n}\pi} = \frac{\pi(1+\text{coſ.}A\frac{m}{n}\pi)}{2n\,\text{ſin.}A.\frac{m}{n}\pi}$

$$= \frac{1}{m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{4n-m} + \frac{1}{4n+m} - \text{\&c.}$$

Ab hujus duplo ſubtrahatur illa ſeries, eritque

$$\frac{\pi}{n\,\text{ſin}\,A.\frac{m}{n}\pi} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \&c.$$

quæ eſt ipſa ſeries §. 12. inventa. Simili autem modo formulæ integrales, quæ pro his ſummis ſunt inventæ ad ſe invicem reducuntur. Cum enim (32) ſit

$$\frac{\pi\,\text{coſ.}A\frac{m}{n}\pi}{n\,\text{ſin.}A\frac{m}{n}\pi} = \int\frac{z^{m-1} - z^{n-m-1}}{1 - z^{n}}dz$$ erit quoque $$\frac{\pi\,\text{coſ.}A\frac{m}{2n}\pi}{2n\,\text{ſin.}A\frac{m}{2n}\pi}$$

$$= \frac{\pi(1+\text{coſ.}A\frac{m}{n}\pi)}{2n\,\text{ſin.}A\frac{m}{n}\pi} = \int\frac{z^{m-1} - z^{2n-m-1}}{1 - z^{2n}}dz$$; ab hujus duplo ſubtrahatur prior, erit $$\frac{\pi}{n\,\text{ſin.}A\frac{m}{n}\pi} =$$

$$\int\frac{2z^{m-1} - 2z^{2n-m-1}}{1 - z^{2n}}dz - \int\frac{z^{m-1} - z^{n-m-1}}{1 - z^{n}}dz =$$

$$\int\frac{z^{m-1} - z^{n+m-1} + z^{n-m-1} - z^{2n-m-1}}{1 - z^{2n}}dz = \int\frac{z^{m-1} + z^{n-m-1}}{1 + z^{n}}dz$$

quæ eſt ipſa integratio §. 29. inventa. Ex quibus perſpicuum eſt, omnia, quæ hactenus ſunt eruta, deduci potuiſſe ex hac ſummatione $\frac{\pi\,\text{coſ.}A\frac{m}{n}\pi}{n\,\text{ſin.}A\frac{m}{n}\pi}$

$$= \int\frac{z^{m-1} - z^{n-m-1}}{1 - z^{n}}dz = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m}$$

+

$+\frac{1}{2n+m}-\frac{1}{3n-m}+\frac{1}{3n+m}-$ &c. Ex qua per differentiationem naſcitur hæc $\frac{\pi\pi}{nn(\text{ſin. A}\frac{m}{n}\pi)^2} =$

$\frac{1}{m^2}+\frac{1}{(n-m)^2}+\frac{1}{(n+m)^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}+$ &c. quæ §. 14. jam eſt inventa.

Problema 5.

40. Invenire differentialia primi, ſecundi, ſequentiumque altiorum ordinum hujus quantitatis

$\frac{\pi \text{ coſ. A.}\frac{m}{n}\pi}{n \text{ ſin. A.}\frac{m}{n}\pi}$ poſito m variabili.

Solutio.

Ponamus brevitatis gratia ſin. $A\frac{m}{n}\pi = x$ & coſ. $A.\frac{m}{n}\pi = y$, erit primo $y = V(1-xx)$; tum vero erit $dx = \frac{\pi dm}{n} y = \frac{\pi y}{n}dm$; & $dy = -\frac{\pi x}{n}dm$. Vocetur quoque quantitas propoſita, cujus differentialia quæruntur,

$\frac{\pi \text{ coſ.A.}\frac{m}{n}\pi}{n \text{ ſin. A.}\frac{m}{n}\pi} = V$, erit $V = \frac{\pi y}{nx}$.

Hinc ergo erit $dV = \frac{\pi(xdy-ydx)}{nxx} = \frac{-\pi\pi dm}{nnxx}$

ob

ob $xx + yy = 1$; ideoque $\frac{dV}{dm} = \frac{-\pi\pi}{nn} \cdot \frac{1}{xx}$ hujus porro ſumatur differentiale, eritque $\frac{ddV}{dm} = + \frac{\pi\pi}{nn} \cdot \frac{2dx}{x^3} = \frac{2\pi^3}{n^3} \cdot \frac{ydm}{x^3}$; ideoque $\frac{d^2V}{dm^2} = \frac{\pi^3}{n^3} \cdot \frac{2y}{x^3}$. Quodſi ſimili modo ſequentia differentialia computentur, ita ſe habebunt.

$$V = + \frac{\pi}{n\,x} \cdot y$$

$$\frac{dV}{dm} = - \frac{\pi^2}{n^2\,x^2} \cdot 1$$

$$\frac{ddV}{dm^2} = + \frac{\pi^3}{n^3\,x^3} \cdot 2y$$

$$\frac{d^3V}{dm^3} = - \frac{\pi^4}{n^4\,x^4} (4yy + 2)$$

$$\frac{d^4V}{dm^4} = + \frac{\pi^5}{n^5\,x^5} (8y^3 + 16y)$$

$$\frac{d^5V}{dm^5} = - \frac{\pi^6}{n^6\,x^6} (16y^4 + 88y^2 + 16)$$

$$\frac{d^6V}{dm^6} = + \frac{\pi^7}{n^7\,x^7} (32y^5 + 416y^3 + 272y)$$

$$\frac{d^7V}{dm^7} = - \frac{\pi^8}{n^8\,x^8} (64y^6 + 1824y^4 + 2886y^2 + 272)$$

&c.

Lex progreſſionis ita ſe habet ut ſi fuerit

$$\frac{d^\nu V}{dm^\nu} = + \frac{\pi^{\nu+1}}{n^{\nu+1}\,x^{\nu+1}} (\alpha y^{\nu-1} + \beta y^{\nu-3} + \gamma y^{\nu-5} + \delta y^{\nu-7} + \varepsilon y^{\nu-9} + \&c.)$$

 futurus

futurus ſit ſequens differentiationis ordo:

$$\frac{d^{\nu+1} V}{d m^{\nu+1}} = \mp \frac{\pi^{\nu+2}}{n^{\nu+2}\, x^{\nu+2}} \left(2\alpha y^{\nu} + (4\beta + (\nu-1)\,\alpha)\, y^{\nu-2} + (6\gamma + (\nu-3)\,\beta)\, y^{\nu-4} + (8\delta + (\nu-5)\,\gamma)\, y^{\nu-6} + (10\epsilon + (\nu-7)\,\delta)\, y^{\nu-8} + \&c.\right)$$

Differentialia igitur cujuscunque ordinis ex præcedentibus determinabuntur. Q. E. J.

Problema. 6.

41. Invenire ſummam hujus ſeriei:

$$\frac{1}{m^{\nu}} + \frac{1}{(m-n)^{\nu}} + \frac{1}{(m+n)^{\nu}} + \frac{1}{(m-2n)^{\nu}} + \frac{1}{(m+2n)^{\nu}} + \frac{1}{(m-3n)^{\nu}}$$

+ &c. ſingulis terminis ſeriei (23) inventæ, ad dignitatem quamcunque elevatis.

Solutio.

Si ponamus ſin $A\,\frac{m}{n}\pi = x$; coſ. $A\,\frac{m}{n}\pi = y$; atque $\frac{\pi y}{n x} = V$ erit ex §. 23:

$$V = \frac{1}{m} + \frac{1}{m-n} + \frac{1}{m+n} + \frac{1}{m-2n} + \frac{1}{m+2n} + \frac{1}{m-3n}$$

+ &c. Quodſi jam poſito m variabili differentialia capiantur, prodibunt ſequentes ſummationes:

$$-\frac{dV}{1\,dm} = \frac{1}{m^2} + \frac{1}{(m-n)^2} + \frac{1}{(m+n)^2} + \frac{1}{(m-2n)^2} + \frac{1}{(m+2n)^2} + \&c.$$

$$+\frac{ddV}{1.2\,dm^2} = \frac{1}{m^3} + \frac{1}{(m-n)^3} + \frac{1}{(m+n)^3} + \frac{1}{(m-2n)^3} + \frac{1}{(m+2n)^3} + \&c.$$

$$-\frac{d^3\,V}{1.2.3\,dm^3}=\frac{1}{m^4}+\frac{1}{(m-n)^4}+\frac{1}{(m+n)^4}+\frac{1}{(m-2n)^4}$$
$$+\frac{1}{(m+2n)^4}+\&c.$$

$$+\frac{d^4\,V}{1.2.3.4\,dm^4}=\frac{1}{m^5}+\frac{1}{(m-n)^5}+\frac{1}{(m+n)^5}+\frac{1}{(m-2n)^5}$$
$$+\frac{1}{(m+2n)^5}+\&c.$$

&c.

Erit argo ſeriei gradus indefiniti propoſitæ:

$$\frac{1}{m^\nu}+\frac{1}{(m-n)^\nu}+\frac{1}{(m+n)^\nu}+\frac{1}{(m-2n)^\nu}+\frac{1}{(m+2n)^\nu}+\frac{1}{(m-3n)^\nu}$$
$$+\ \&c.$$

ſumma $=\frac{\pm d^{\nu-1}\,V}{1.2.3.\ldots\ldots(\nu-1)\,dm^{\nu-1}}$. At problemate præcedente valorem ipſius $\frac{d^{\nu-1}\,V}{dm^{\nu-1}}$ exhibuimus; quamobrem quoque ſummæ harum ſerierum poteſtatum poterunt definiri.

Q. E. J.

Problema 7.

42. Sinum anguli cujuscunque $\frac{m}{n}\pi$ per productum ex infinitis factoribus exhibere.

Solutio.

Cum ſit $\frac{\pi\,\text{coſ.A.}\frac{m}{n}\pi}{n\,\text{ſin. A.}\frac{m}{n}\pi}=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}$ $+$ &c. tractetur m uti quantitas variabilis, ac multiplicetur ubique

ubique per dm, erit $\frac{\pi\, dm}{n}$ coſ. $A\frac{m}{n}\pi = d.$ ſin. $A\frac{m}{n}\pi$

& hancobrem erit $\frac{\pi\, dm \text{ coſ. } A.\frac{m}{n}\pi}{n \text{ ſin } A\frac{m}{n}\pi} = \frac{d. \text{ ſin. } A\frac{m}{n}\pi}{\text{ſin. } A\frac{m}{n}\pi}$

$= \frac{dm}{m} - \frac{dm}{n-m} + \frac{dm}{n+m} - \frac{dm}{2n-m} + \frac{dm}{2n+m} -$ &c.

unde integratione utrinque abſoluta erit l ſin. $A.\frac{m}{n}\pi =$ $lm + l(n-m) + l(n+m) + l(2n-m) + l(2n+m) +$ &c. $+ C$
Conſtans C ita eſſe debet comparata, ut facto $m = \frac{1}{2}n$, logarithmus ſinus fiat $= 0$, quippe quo caſu habetur ſinus totus. Hoc ergo modo conſtante C determinata erit l ſin. $A.\frac{m}{n}\pi = l\frac{2m}{n} + l\frac{2n-2m}{n} + l\frac{2n+2m}{3n} + l\frac{4n-2m}{3n} +$ $l\frac{4n+2m}{5n} +$ &c. Unde ſi tranſeamus ad numeros, habebimus:

$$\text{ſin. A}\frac{m}{n}\pi = \frac{2m}{n} \cdot \frac{2n-2m}{n} \cdot \frac{2n+2m}{3n} \cdot \frac{4n-2m}{3n} \cdot \frac{4n+2m}{5n} \cdot \text{\&c.}$$

Vel ſi binos factores in ſe ducamus, erit

$$\text{ſiu. A}\frac{m}{n}\pi = \frac{2m}{n} \cdot \frac{4nn-4mm}{3nn} \cdot \frac{16nn-4mm}{15nn} \cdot \frac{36nn-4mm}{35nn} \cdot \text{\&c.}$$

Q. E. J.

Coroll. 1.

43. Si loco $2m$ ſcribamus m habebimus:

$$\text{ſin. A}\frac{m}{2n}\pi = \frac{m}{n} \cdot \frac{2n-m}{n} \cdot \frac{2n+m}{3n} \cdot \frac{4n-m}{3n} \cdot \frac{4n+m}{5n} \cdot \text{\&c.}$$

$$\text{ſive ſin A.}\frac{m}{2n}\pi = \frac{m}{n} \cdot \frac{4nn-mm}{4nn-nn} \cdot \frac{16nn-mm}{16nn-nn} \cdot \frac{36nn-mm}{36nn-nn} \cdot \text{\&c.}$$

Coroll. 2.

Coroll. 2

44. Quia eſt ſin A $\frac{m}{2n}\pi = $ coſ. A $\frac{(n-m)}{2n}\pi$, erit ſi $n - m$ ſcribamus loco m, ex ſerie inventa:

$$\text{coſ. A}\ \frac{m}{2n}\pi = \frac{n-m}{n} \cdot \frac{n+m}{n} \cdot \frac{3n-m}{3n} \cdot \frac{3n+m}{3n} \cdot \text{\&c.}$$

$$\text{ſive coſ. A}\ \frac{m}{2n}\pi = \frac{nn-mm}{nn} \cdot \frac{9nn-mm}{9nn} \cdot \frac{25nn-mm}{25nn} \cdot \text{\&c.}$$

Coroll. 3.

45. Quoniam eſt ſin A $\frac{m}{n}\pi = 2$ ſin A $\frac{m}{2n}\pi$. coſ. A $\frac{m}{2n}\pi$ ſi ſin A. $\frac{m}{n}\pi$ dividamus per 2 ſin A $\frac{m}{2n}\pi$, habebimus

$$\text{coſ. A}\frac{m}{2n} = \frac{2n-2m}{2n-m} \cdot \frac{2n+2m}{2n+m} \cdot \frac{4n-2m}{4n-m} \cdot \frac{4n+2m}{4n+m} \cdot$$

&c. & diviſo ſin A $\frac{m}{n}\pi$ per 2 coſ. *A* $\frac{m}{2n}\pi$ habebimus:

$$\text{ſin}\ A\ \frac{m}{2n}\pi = \frac{m}{n-m} \cdot \frac{2n-2m}{n+m} \cdot \frac{2n+2m}{3n-m} \cdot \frac{4n-2m}{3n+m} \cdot \frac{4n+2m}{5n-m} \cdot$$

&c.

Coroll. 4.

46. Duplices iſtæ ſinuum & coſinuum expreſſiones inter ſe æquatæ dabunt.

$$1 = \frac{nn}{nn-mm} \cdot \frac{4nn-4mm}{4nn-mm} \cdot \frac{9nn}{9nn-mm} \cdot \frac{16nn-4mm}{16nn-mm} \cdot \frac{25nn}{25nn-mm}\ \text{\&c.}$$

Coroll. 5.

47. Si n capiatur infinitum, ſeu m infinite parvum erit,

ſin A. $\frac{m}{2n}\pi = \frac{m}{2n}\pi$, hocque caſu ex utraque ſerie naſcitur idem valor ipſius π a Walliſio datus.

$$\pi = 2.\ \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ 10.\ 10.}{1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 9.\ \ 9.\ 11.}\ \&c.$$

Lemma 4.

48. Valor hujus producti ex infinitis factoribus conſtantis

$$\frac{a\ (c+b)\ (a+k)\ (c+b+k)\ (a+2k)\ (c+b+2k)\ \&c.}{b\ (c+a)\ (b+k)\ (c+a+k)\ (b+2k)\ (c+a+2k)\ \&c.}$$

eſt $= \frac{\int z^{c-1}\, dz\ (1-z^k)^{\frac{b-k}{k}}}{\int z^{c-1}\, dz.(1-z^k)^{\frac{a-k}{k}}}$, ſi poſt utramque integrationem ponatur $z = 1$.

Problema 8.

49. Sinum anguli $\frac{m}{2n}\pi$ per formulas integrales exprimere.

Solutio.

Cum ſit ſin A. $\frac{m}{2n}\pi = \frac{m}{n}\ .\ \frac{2n-m}{n}\ .\ \frac{2n+m}{3n}\ .\ \frac{4n-m}{3n}$ &c. per §. 43. comparetur hoc productum infinitum cum lemmate præcedente, eritque $a = m$; $b = n$; $k = 2n$; & $c + m = n$ vel $c + n = 2n - m$, utrumque dat $c = n - m$. Hinc igitur fiet

$$\text{ſin A}\ \frac{m}{2n}\pi = \frac{\int z^{n-m-1}\, dz\,(1-z^{2n})^{-\frac{1}{2}}}{\int z^{n-m-1}\, dz\ (1-z^{2n})^{\frac{-2n+m}{2n}}}$$

ſi poſt utramque integrationem ita inſtitutam ut integralia

eva-

evaneſcant poſito $z = 0$, ponatur $z = 1$.

Cum vero etiam ſit per §. 45. $2 \sin A \frac{m}{2n}\pi =$
$\frac{2m}{n-m} \cdot \frac{2n-2m}{n+m} \cdot \frac{2n+2m}{3n-m} \cdot \frac{4n-2m}{3n+m} \cdot$ &c. erit comparatione cum lemmate inſtituta: $a = 2m$; $b = n-m$; $c = n-m$; & $k = 2n$, unde obtinebitur.

$$2 \sin A.\frac{m}{2n}\pi = \frac{\int z^{n-m-1} dz\, (1-z^{2n})^{\frac{-n-m}{2n}}}{\int z^{n-m-1} dz\, (1-z^{2n})^{\frac{m-n}{n}}}$$

ſi poſt integrationes ponatur $z = 1$. Q. E. J.

Coroll. 1.

50. Sequentes ergo nanciſcimur diverſarum formularum integralium comparationes.

$$\sin A.\frac{m}{2n}\pi.\int\frac{z^{n-m-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}} = \int\frac{z^{n-m-1}dz}{\sqrt{(1-z^{2n})}}$$

&

$$2\sin A.\frac{m}{2n}\pi.\int\frac{z^{n-m-1}dz}{(1-z^{2n})^{\frac{n-m}{n}}} = \int\frac{z^{n-m-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}}$$

Coroll. 2.

51. Tum vero ſine ſinus ratione habita inſtitui poteſt iſta integralium comparatio.

$$2\int\frac{z^{n-m-1}dz}{(1-z^{2n})^{\frac{n-m}{n}}} : \int\frac{z^{n-m-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}} = \int\frac{z^{n-m-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}} : \int\frac{z^{n-m-1}dz}{\sqrt{(1-z^{2n})}}$$

Co-

Coroll. 3.

52. Ponamus eſſe $m = 1$ & $n = 1$; erit ſin $A \frac{m}{2n}\pi = 1$ atque comparationes ita ſe habebunt

$$\int \frac{dz}{z\sqrt{(1-zz)}} = \int \frac{dz}{z\sqrt{(1-zz)}} \quad \& \quad 2\int \frac{dz}{z} = \int \frac{dz}{z(1-zz)}$$

quarum æquationum poſteriori duo ſpatia hyperbolica infinita inter ſe comparantur.

Coroll. 4.

53. Ponamus eſſe $m = 2$ & $n = 3$ erit ſin $A \frac{m}{2n}\pi = \frac{\sqrt{3}}{2}$ unde orientur ſequenter comparationes.

$$\frac{\sqrt{3}}{2}\int \frac{dz}{(1-z^6)^{\frac{2}{3}}} = \int \frac{dz}{\sqrt{(1-z^6)}} \text{ \&c. } \sqrt{3}.\int \frac{dz}{(1-z^6)^{\frac{1}{3}}} = \int \frac{dz}{(1-z^6)^{\frac{5}{6}}}$$

ex his naſcitur iſta proportio:

$$\tfrac{1}{2}\int \frac{dz}{(1-z^6)^{\frac{2}{3}}} : \int \frac{dz}{(1-z^6)^{\frac{1}{3}}} = \int \frac{dz}{(1-z^6)^{\frac{1}{2}}} : \int \frac{dz}{(1-z^6)^{\frac{5}{6}}}$$

Coroll. 5.

54. Sit $m = 1$; $n = 2$, ut ſit ſin $A \frac{m}{2n}\pi = \frac{1}{\sqrt{2}}$ erit

$$\frac{1}{\sqrt{2}}\int \frac{dz}{(1-z^4)^{\frac{3}{4}}} = \int \frac{dz}{(1-z^4)^{\frac{1}{2}}}; \quad \& \sqrt{2}.\int \frac{dz}{(1-z^4)^{\frac{1}{2}}} = \int \frac{dz}{(1-z^4)^{\frac{3}{4}}}$$

quæ duæ æquationes inter ſe congruunt.

Coroll. 6.

55. Sit $m = 1$, $n = 3$, ut ſit ſin $A.\frac{m}{2n}\pi = \frac{1}{2}$ erit:

$$\tfrac{1}{2}\int \frac{zdz}{(1-z^6)^{\frac{5}{6}}} = \int \frac{zdz}{(1-z^6)^{\frac{1}{2}}} \quad \& \quad \int \frac{zdz}{(1-z^6)^{\frac{2}{3}}} = \int \frac{zdz}{(1-z^6)^{\frac{2}{3}}}$$

quarum

quarum poſterior eſt identica, prior autem dat

$\int\frac{zdz}{(1-z^6)^{\frac{5}{6}}} = 2\int\frac{zdz}{\sqrt{(1-z^6)}}$, poſito poſt integrationem $z =$ 1, quæ conditio ſemper adjuncta eſt intelligenda.

Problema. 9.

56. Expreſſiones infinitas, quas pro coſinu anguli $\frac{m}{2n}\pi$ invenimus, ad formulas integrales reducere.

Solutio.

Primum §. 44. invenimus eſſe coſ. A. $\frac{m}{2n}\pi =$

$\frac{n-m}{n} \cdot \frac{n+m}{n} \cdot \frac{3n-m}{3n} \cdot \frac{3n+m}{3n} \cdot$ &c. quæ expreſſio cum lemmate (48) comparata dat $a = n - m$; $b = n$; $c = m$ & $k = 2n$, quibus ſubſtitutis oritur

$$\text{coſ.A.}\,\frac{m}{2n}\pi = \frac{\int z^{m-1}\,dz\,(1-z^{2n})^{-\frac{1}{2}}}{\int z^{m-1}\,dz\,(1-z^{2n})^{\frac{-n-m}{2n}}}$$

Deinde §. 45. vidimus eſſe coſ A $\frac{m}{2n}\pi =$

$\frac{2n-2m}{2n-m} \cdot \frac{2n+2m}{2n+m} \cdot \frac{4n-2m}{4n-m} \cdot \frac{4n+2m}{4n+m} \cdot$ &c.

qua expreſſione cum lemmate comparata reperietur $a = 2n - 2m$, $b = 2n - m$; $c = 3m$; & $k = 2n$; unde erit:

$$\text{coſ}\,A\,\frac{m}{2n}\pi = \frac{\int z^{3m-1}\,dz\,(1-z^{2n})^{-\frac{m}{2n}}}{\int z^{3m-1}\,dz\,(1-z^{2n})^{-\frac{m}{n}}}$$

 ſi poſt

ſi poſt integrationem ponatur $z = 1$. Q. E. J.

Coroll. 1.

57. Hinc iterum ſinus anguli $\frac{m}{2n}\pi$ exprimi poteſt ponendo $n-m$ loco m, prior quidem expreſſio dat eam ipſam quam jam invenimus, at ex poſteriori naſcitur.

$$\text{ſin } A\,\frac{m}{2n}\pi = \frac{\int z^{3n-3m-1}\,dz\,(1-z^{2n})^{-\frac{n+m}{2n}}}{\int z^{3n-3m-1}\,dz\,(1-z^{2n})^{-\frac{n+m}{n}}}$$

Coroll. 2.

58. Quemadmodum tres expreſſiones pro ſinu habemus, ita ad duas expreſſiones pro coſinu inventas tertia accedet ex ſecunda expreſſione ſinus, quæ dabit

$$2\,\text{coſ} A\,\frac{m}{2n}\pi = \frac{\int z^{m-1}\,dz\,(1-z^{2n})^{-\frac{2n+m}{2n}}}{\int z^{m-1}\,dz\,(1-z^{2n})^{-\frac{m}{n}}}$$

Coroll. 3.

59. Hinc igitur innumerabilia paria formularum integralium caſu quo $z = 1$ inter ſe comparari poterunt, hæque comparationes pendebunt a multiſectione anguli.

Problema. 10.

60. Invenire expreſſiones integrales, quæ caſu $z = 1$ tangentem anguli $\frac{m}{2n}\pi$ exhibeant.

Solutio.

Cum tangens anguli ſit quotus ex diviſione ſinus per coſinum ortus erit ex §. § 43. & 44:

tang.

$$\text{tang. } A. \frac{m}{2n}\pi = \frac{m}{n-m} \cdot \frac{2n-m}{n+m} \cdot \frac{2n+m}{3n-m} \cdot \frac{4n-m}{3n+m} \cdot$$

&c. comparetur hæc expreſſio cum lemmate (48), eritque $a = m$; $b = n - m$; $c = n$, & $k = 2n$, unde orietur

$$\text{tang A. } \frac{m}{2n}\pi = \frac{\int z^{n-1} dz \left(1-z^{2n}\right)^{\frac{-n-m}{2n}}}{\int z^{n-1} dz \left(1-z^{2n}\right)^{\frac{m-m}{2n}}}$$

poſito poſt utramque integrationem $z = 1$.

Deinde ex §. 45 elicitur pro tangente iſta expreſſio

$$\text{tang.A } \frac{m}{2n}\pi = \frac{m}{n-m} \cdot \frac{2n-m}{n+m} \cdot \frac{2n+m}{3n-m} \cdot \frac{4n-m}{3n+m} \cdot \text{ \&c.}$$

ex qua eadem expreſſio integralis, quæ ante invenitur. Q. E. J.

Coroll. 1.

61. Ponamus $m = 2$ & $n = 3$ erit tang A $\frac{m}{2n}\pi = \sqrt{3}$, hincque

$$\sqrt{3}. \int \frac{zz\, dz}{(1-z^6)^{\frac{2}{3}}} = \int \frac{zz\, dz}{(1-z^6)^{\frac{5}{6}}}.$$

ſi ponamus $z^3 = v$ erit $zz\, dz = \frac{1}{3} dv$, ac proinde

$$\int \frac{dv \sqrt{3}}{(1-v^2)^{\frac{2}{3}}} = \int \frac{dv}{(1-v^2)^{\frac{5}{6}}}$$

Coroll. 2.

62. Si generaliter ponamus $z^n = v$ habebitur

$$\text{tang A} \frac{m}{2n}\pi = \frac{\int dv \ (1-vv)^{\frac{-n-m}{2n}}}{\int dv \ (1-vv)^{\frac{m-2n}{2n}}} \quad \text{ſive}$$

$$\text{tang A. } \frac{m}{2n}\pi. \int \frac{dv}{(1-vv)^{\frac{2n-m}{2n}}} = \int \frac{dv}{(1-vv)^{\frac{n+m}{2n}}}$$

Coroll. 3.

Coroll. 3.

63. Sit $m = 1$, $n = 2$ erit tang $\frac{m}{2n}\pi = 1$ hincque

$$\int \frac{dv}{(1-vv)^{\frac{3}{4}}} = \int \frac{dv}{(1-vv)^{\frac{3}{4}}}$$

quæ æquatio identica inſervit veritati calculi comprobandæ.

Scholion.

64. Plures hujus generis comparationes inſtitui poterunt, ſi in ſubſidium vocentur theoremata circa comparationes formularum integralium alibi a me demonſtrata (Tom. Comment. X.) unde quædam inſtar lemmatum depromam.

Lemma 5.

65. Si poſt integrationes ubique ponatur $z = 1$ erit

$$\int \frac{z^{a-1}\,dz}{(1-z^b)^{1-c}} \cdot \int \frac{z^{a+bc-1}\,dz}{(1-z^b)^{1-\gamma}} = \int \frac{z^{a-1}\,dz}{(1-z^b)^{1-\gamma}} \cdot \int \frac{z^{a+b\gamma-1}\,dz}{(1-z^b)^{1-c}}$$

Lemma 6.

66. Si poſt integrationes ponatur $z = 1$ erit

$$\frac{b+1}{c+1} = \frac{\int z^{b(\frac{1}{2}-k)-1}\,dz\,(1-z^b)^c \cdot \int z^{b(\frac{3}{2}+c-k)-1}\,dz\,(1-z^b)^{-\frac{1}{2}+k}}{\int z^{b(\frac{1}{2}-k)-1}\,dz\,(1-z^b)^b \cdot \int z^{b(\frac{3}{2}+b+k)-1}\,dz\,(1-z^b)^{-\frac{1}{2}-k}}$$

Lemma 7.

67. Si poſt integrationes ponatur $z = 1$ erit

$$\frac{c}{a} = \frac{\int z^{a-1}\,dz\,(1-z^b)^{-\frac{1}{2}+k} \cdot \int z^{a+(\frac{1}{2}+k)b-1}\,dz\,(1-z^b)^{-\frac{1}{2}-k}}{\int z^{c-1}\,dz\,(1-z^b)^{-\frac{1}{2}-k} \cdot \int z^{c+(\frac{1}{2}-k)b-1}\,dz\,(1-z^b)^{-\frac{1}{2}+k}}$$

Lemma 8.

Lemma 8.

68. Si poſt integrationes ponatur $z = 1$ erit

$$\frac{(a+1)(a-k+1)}{(c+1)(c+k+1)} = \frac{\int z^{b(1+k)-1} dz (1-z^b)^c \,.\, \int z^{b(1-k)-1} dz (1-z^b)^{c+k}}{\int z^{b(1-k)-1} dz (1-z^b)^a \,.\, \int z^{b(1+k)-1} dz (1-z^b)^{a-k}}$$

THEOREMA. 3.

69. Si poſt integrationes ponatur $z = 1$ erit

$$\operatorname{coſ} A \frac{m}{2n}\pi = \frac{\int \frac{z^{m-1} dz}{(1-z^{2n})^{\frac{1}{2}}} \cdot \int \frac{z^{n-1} dz}{(1-z^{2n})^{1-c}}}{\int \frac{z^{m-1} dz}{(1-z^{2n})^{1-c}} \cdot \int \frac{z^{m+2nc-1} dz}{(1-z^{2n})^{\frac{n+m}{2n}}}}$$

Demonſtratio.

Si enim in lemmate 5. ponamus $a = m$; $b = 2n$; & $\gamma = \frac{n-m}{2n}$ fit $\int \frac{z^{a-1} dz}{(1-z^b)^{1-\gamma}} = \int \frac{z^{m-1} dz}{(1-z^{2n})^{\frac{n+m}{2n}}}$

At per §. 56. eſt $\int \frac{z^{m-1} dz}{(1-z^{2n})^{\frac{n+m}{2n}}} = \frac{1}{\operatorname{coſ} A \frac{m}{2n}\pi} \int \frac{z^{m-1} dz}{(1-z^{2n})^{\frac{1}{2}}}$

qui valor in lemmate ſubſtitutus dabit æqualitatem, quam demonſtrari oportebat.

Coroll. 1.

70. Ineſt in hac æqualitate exponens indefinitus c, quem pro arbitrio determinare licet; ſit igitur $c = \frac{1}{2}$; & quia numerator & denominator factorem habent communem, erit:

$$\operatorname{cof} A \frac{m}{2n}\pi . \int \frac{z^{n+m-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}} = \int \frac{z^{n-1}dz}{\sqrt{(1-z^{2n})}}$$

Coroll. 2.

71. Si in formula $\int \frac{z^{n-1}dz}{\sqrt{(1-z^{2n})}}$ ponamus $z^n = v$ abit ea in $\frac{1}{n}\int \frac{dv}{\sqrt{(1-vv)}}$ cujus integrale poſito $z = 1$ ſeu $v = 1$ erit $\frac{\pi}{2n}$. Hanc obrem erit

$$\int \frac{z^{n+m-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}} = \frac{\pi}{2n \operatorname{cof} \text{A} \frac{m}{2n}\pi} \text{ poſito } z = 1.$$

Coroll. 3.

72. Si ponamus $z = \frac{u}{(1+u^{2n})^{\frac{1}{2n}}}$, ut loco variabilis z introducamus u; eritque $z = o$ ſi $u = o$, at fit $z = 1$ ſi $u = \infty$. Quamobrem facta ſubſtitutione erit

$$\int \frac{u^{n+m-1}du}{1+u^{2n}} = \frac{\pi}{2n \operatorname{cof} A \frac{m}{2n}\pi}$$

poſito poſt integrationem $u = \infty$.

Coroll. 4.

73. Si in §. 29. loco n ponamus $2n$ erit $\frac{\pi}{2n \operatorname{ſin} \text{A} \frac{m}{2n}\pi}$

$= \int$

$= \int \frac{z^{m-1} + z^{2n-m-1}}{1+z^{2n}} dz$ ſi poſt integrationem fiat $z = 1$.

Quod ſi ergo pro m ſcribatur $n-m$ fiet

$\frac{\pi}{2n \operatorname{coſ} A \frac{m}{2n}\pi} = \int \frac{z^{n-m-1} + z^{n+m-1}}{1+z^{2n}} dz$, poſito poſt integrationem $z = 1$, quod ergo integrale æquatur huic

$$\int \frac{u^{n+m-1}\, du}{1+u^{2n}} \quad \text{ſi ponatur } u = \infty.$$

THEOREMA. 4.

74. Si poſt integrationes ponatur $z = 1$, erit

$$2 \operatorname{coſ} A. \frac{m}{2n}\pi. \int \frac{z^{2m-1}\, dz}{(1-z^{2n})^{\frac{m}{n}}} = \int \frac{z^{2n-m-1}\, dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}$$

Demonſtratio.

In §. 58. nacti ſumus hanc coſinus expreſſionem

$$2 \operatorname{coſ} A \frac{m}{2n}\pi = \frac{\int z^{m-1}\, dz\, (1-z^{2n})^{-\frac{2n+m}{2n}}}{\int z^{m-1}\, dz\, (1-z^{2n})^{-\frac{m}{n}}}$$

Si jam in lemmate 5, faciamus $a = m, b = 2n, c = \frac{m}{2n}$ & $\gamma = \frac{n-m}{n}$, duæ lemmatis formulæ integrales in has, quæ $2 \operatorname{coſ} A \frac{m}{2n}\pi$ exprimunt, transmutantur, quarum loco ſi ſcribatur $2 \operatorname{coſ} A. \frac{m}{2n}\pi$, prodibit

2 coſ

$$2\operatorname{cof}A.\frac{m}{2n}\pi.\int\frac{z^{2m-1}dz}{(1-z^{2n})^{\frac{m}{n}}}=\int\frac{z^{2n-m-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}$$

Q. E. D.

Coroll. 1.

75. Si hinc in lemmate ſexto ponatur $b=2n$; $c=\frac{-m}{n}$ & $k=\frac{n-2m}{2n}$ formula $\int z^{b(\frac{1}{2}-k)-1}dz\ (1-z^{b})^{c}$ abit in hanc $\int\frac{z^{2m-1}dz}{(1-z^{2n})^{\frac{m}{n}}}$ cujus loco ſi ſcribamus $\frac{1}{2\operatorname{cof}A\frac{m}{2n}\pi}\int\frac{z^{2n-m-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}$ faciamusque $b=0$, obtinebimus hanc reductionem

$$2\operatorname{cof}A\frac{m}{2n}\pi=\frac{\int\frac{z^{2n-m-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}}{\int\frac{z^{2n-2m-1}dz}{(1-z^{2n})^{\frac{n-m}{n}}}}\ \text{ſeu}\ \int\frac{z^{2m-1}dz}{(1-z^{2n})^{\frac{m}{n}}}=\int\frac{z^{2n-2m-1}dz}{(1-z^{2n})^{\frac{n-m}{n}}}$$

Coroll. 2.

76. Si ponamus $m=2$ & $n=3$ erit $\operatorname{cof}A\frac{m}{2n}\pi=\frac{1}{2}$ unde æquatio theorematis dabit:

$\int\frac{z^{3}dz}{(1-z^{6})^{\frac{2}{3}}}=\int\frac{z^{3}dz}{(1-z^{6})^{\frac{2}{3}}}$, at æquatio coroll. præced. dabit

$\int$

$\int\frac{z\,dz}{(1-z^6)^{\frac{1}{3}}} = \int\frac{z^3\,dz}{(1-z^6)^{\frac{2}{3}}}$ ſeu poſito z loco zz hanc

$\int\frac{dz}{(1-z^3)^{\frac{1}{3}}} = \int\frac{z\,dz}{(1-z^3)^{\frac{2}{3}}}$, poſito $z = 1$.

Coroll. 3.

77. Sit $m = 1$, & $n = 2$, fiet coſ $A\frac{m}{2n}\pi = \frac{1}{\sqrt{2}}$, ideoque

$$\int\frac{z\,dz\sqrt{2}}{(1-z^4)^{\frac{1}{2}}} = \int\frac{zz\,dz}{(1-z^4)^{\frac{3}{4}}} = \frac{\pi}{2\sqrt{2}} \text{ ob } \int\frac{z\,dz}{\sqrt{(1-z^4)}} = \frac{\pi}{4}.$$

Ex coroll. 1. vero erit: $\int\frac{z\,dz\sqrt{2}}{(1-z^4)^{\frac{1}{2}}} = \int\frac{zz\,dz}{(1-z^4)^{\frac{2}{4}}}$, quæ eſt eadem æqualitas.

THEOREMA. 5.

98. Si poſt integrationes ponatur $z = 1$ erit

$$\text{tang.}\frac{m}{2n}\pi \cdot \int\frac{z^{n-1}\,dz}{(1-z^{2n})^{\frac{2n-m}{2n}}} \cdot \int\frac{z^{2n-m-1}\,dz}{(1-z^{2n})^{1-\gamma}} = \int\frac{z^{n-1}\,dz}{(1-z^{2n})^{1-\gamma}} \cdot \int\frac{z^{n+2n\gamma-1}\,d\dots}{(1-z^{2n})^{\frac{n+\dots}{2n}}}$$

Demonſtratio.

In §. 60. invenimus eſſe $\int\frac{z^{n-1}\,dz}{(1-z^{2n})^{\frac{n+m}{2n}}} = \text{tang}\,A\frac{m}{2n}\pi$

$\int\frac{z^{n-1}\,dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}$. Fiat jam in lemmate 5; $a = n$; $b = 2n$; & $c = \frac{n-m}{2n}$, atque facta ſubſtitutione erit

tang.

$$\text{tang.}A\tfrac{m}{2n}\pi \int\frac{z^{n-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}\cdot\int\frac{z^{2n-m-1}dz}{(1-z^{2n})^{1-\gamma}}=\int\frac{z^{n-1}dz}{(1-z^{2n})^{1-\gamma}}\cdot\int\frac{z^{n+2n\gamma-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}}$$

Q. E. D.

Coroll. 1.

79. Si ponatur $\gamma = 1$, ob duo membra integrabilia fiet

$$\frac{n}{2n-m}\ \text{tang.}A\,\frac{m}{2n}\pi.\int\frac{z^{n-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}=\int\frac{z^{3n-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}}=$$

$$\frac{n}{2n-m}\int\frac{z^{n-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}}$$ hancobrem erit:

$$\text{tang}\,A.\frac{m}{2n}\pi.\int\frac{z^{n-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}=\int\frac{z^{n-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}}$$

quæ eſt ipſa in §. 60. inventa.

Coroll. 2.

80. Sit $\gamma=\frac{m}{2n}$ erit $\text{tang}\,A.\frac{m}{2n}\pi.\int\frac{z^{2n-m-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}=\int\frac{z^{n+m-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}}$

ac, ſi ponatur $\gamma=\frac{1}{2}$, ingredietur quadratura circuli, eritque

$$\int\frac{z^{2n-m-1}dz}{\sqrt{(1-z^{2n})}}\cdot\int\frac{z^{n-1}dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}=\frac{\pi}{2n\,\text{tang}\,A\,\frac{m}{2n}\pi}\int\frac{z^{2n-1}dz}{(1-z^{2n})^{\frac{n+m}{2n}}}=$$

$$\frac{\pi}{2n(n-m)\text{tang}\,A\frac{m}{2n}\pi};\ \text{ſeu}\ \frac{\pi\,\text{tang}\,A\frac{m}{2n}\pi}{2mn}=\int\frac{z^{n-1}dz}{(1-z)^{\frac{n+m}{2n}}}\cdot\int\frac{z^{n+m-1}dz}{\sqrt{(1-z^{2n})}}$$

Coroll.

Coroll. 3.

81. Cum igitur ſit $\frac{\pi}{2mn}$ tang A. $\frac{m}{2n}\pi = \int \frac{z^{n-1}\,dz}{(1-z^{2n})^{\frac{n+m}{2n}}} \cdot \int \frac{z^{n+m-1}\,dz}{\sqrt{(1-z^{2n})}}$

atque ex §.60. ſit $\int \frac{z^{n-1}\,dz}{(1-z^{2n})^{\frac{n+m}{2n}}} =$ tang A $\frac{m}{2n}\pi. \int \frac{z^{n-1}\,dz}{(1-z^{2n})^{\frac{2n-m}{2n}}}$

erit $\frac{\pi}{2mn} = \int \frac{z^{n-1}\,dz}{(1-z^{2n})^{\frac{2n-m}{2n}}} \cdot \int \frac{z^{n+m-1}\,dz}{\sqrt{(1-z^{2n})}}$. Productum ergo harum duarum formularum integralium caſu quo $z=1$ per peripheriam circuli exhiberi poteſt.

Coroll. 4.

82. Si $m=1$ & $n=1$ erit ex coroll. præcedente $\frac{\pi}{2} = \int \frac{dz}{\sqrt{(1-zz)}} \cdot \int \frac{z\,dz}{\sqrt{(1-zz)}} = \frac{\pi}{2} \cdot \left(1-\sqrt{(1-zz)}\right)$ quo caſu ſi fiat $z=1$, æqualitas ſponte perſpicitur.

Coroll. 5.

83. Sit $m=1$, & $n=2$, erit tang. A. $\frac{m}{2n}\pi = 1$, hinc ex coroll. 2 erit $\frac{\pi}{4} = \int \frac{z\,dz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{zz\,dz}{\sqrt{(1-z^4)}}$. ex tertio autem oritur $\frac{\pi}{4} = \int \frac{z\,dz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{zz\,dz}{\sqrt{(1-z^4)}}$. quæ duæ æquationes inter ſe congruunt.

Coroll. 6.

Coroll. 6.

84. Sit $m = 2$ & $n = 3$, erit tang. A $\frac{m}{2n}\pi = \sqrt{3}$: hinc ex coroll. 2. oritur $\frac{\pi}{4\sqrt{3}} = \int \frac{zz\,dz}{(1-z^6)^{\frac{5}{3}}} \cdot \int \frac{z^4\,dz}{\sqrt{(1-z^6)}}$: ex tertio vero naſcitur hæc æquatio $\frac{\pi}{12} = \int \frac{zz\,dz}{(1-z^6)^{\frac{2}{3}}} \cdot \int \frac{z^4\,dz}{\sqrt{(1-z^6)}}$.

Scholion.

85. Hujusmodi theorematum ex formulis integralibus pro ſinu, coſinu & tangente inventis ope lemmatum 5, 6, 7 & 8 tanta multitudo deduci poteſt, ut iis capiendis integrum volumen non ſufficeret. Aperto autem fonte quilibet, quantum libuerit, inde haurire poterit. Complures quidem occurrunt caſus, uti vidimus, quibus vel ad æquationes identicas vel ejusmodi, quæ facile eo reducuntur, pervenitur, hique caſus veritatem reliquorum theorematum eo magis confirmant, in quibus ratio æqualitatis non perſpicitur. Sic in æquatione §. 30. ſi ponatur $m = 0$ & $n = 1$, fiet tang A $\frac{m}{2n}\pi = \frac{m}{2n}\pi$, eo quod tangens arcus evaneſcentis ipſi arcui æquatur; hinc igitur fiet $\frac{\pi\pi}{4} = \int \frac{dz}{\sqrt{(1-zz)}} \cdot \int \frac{dz}{\sqrt{(1-zz)}}$, cujus veritas, cum ſit $\int \frac{dz}{\sqrt{(1-zz)}} = \frac{\pi}{2}$ caſu quo $z = 1$ ſponte apparet. Ceterum hujusmodi formularum integralium, quæ neque integrari, neque ad ſe mutuo reduci poſ-

ſunt

ſunt, comparationes eo magis ſunt notatu dignæ, quo minus via ad eas comprobandas patere videatur. Sic primum hujus generis theorema, ad quod jam pridem fui deductus, ſimplicitate ſe commendabat, quo inveni eſſe productum harum duarum formularum integralium $\int \frac{dz}{\sqrt{(1-z^4)}}$ & $\int \frac{z^2 dz}{\sqrt{(1-z^4)}}$ quarum altera arcum, altera ordinatam in curva elaſtica exprimit, caſu quo $z = 1$, æquale areæ circuli cujus diameter ſit $= 1$.

Num. III.

Ejusdem

De Inventione Integralium ſi poſt integrationem variabili quantitati determinatus valor tribuatur.

Lemma 1.

1. Invenire ſummam ſeriei recurrentis Tab II.
$A + B + C + D + \ldots\ldots + P$ in qua quilibet terminus ex duobus præcedentibus ita formatur ut ſit $C = mB + nA$, $D = mC + nB$; &c.

Solutio.

Quo ſolutio latius pateat, multiplicemus ſingulos terminos per terminos progreſſionis geometricæ, ut habeamus hanc ſeriem: